

$L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ DOES NOT EMBED IN $L_{2,\mathbb{Z}}$

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ABSTRACT. For a commutative ring R with unit we investigate the embedding of tensor product algebras into the Leavitt algebra $L_{2,R}$. We show that the tensor product $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ does not embed in $L_{2,\mathbb{Z}}$ (as a unital $*$ -algebra). We also prove a partial non-embedding result for the more general $L_{2,R} \otimes L_{2,R}$. Our techniques rely on realising Thompson's group V as a subgroup of the unitary group of $L_{2,R}$.

1. INTRODUCTION

Since the beginning of the study of Leavitt path algebras ([5, 8]), it has been known that there is a strong connection between Leavitt path algebras and graph C^* -algebras. In particular, some of the remarkable classification results for purely infinite C^* -algebras have been shown to have partial algebraic analogues for Leavitt path algebras ([2, 3, 20]). In [12] and this paper the authors continue the tradition by studying possible algebraic analogues of Kirchberg's celebrated embedding Theorem: All separable, exact C^* -algebras embed into \mathcal{O}_2 .

The first problem to overcome is to make sense of what is meant by the algebraic analogue. We certainly want to replace \mathcal{O}_2 with $L_{2,R}$, but then have to decide how to translate the conditions of being separable and exact. Since $L_{2,R}$ has a countable basis as an R -module, it is only possible to embed R -algebras which have a countable basis into it, so we choose that as our algebraic version of separable. What to replace exactness with seems a more prickly question. Exactness of C^* -algebras has a multitude of equivalent but very different looking definitions [10, Theorem IV.3.4.18], so we are forced to make a choice. One definition, where the name exact comes from, says that a C^* -algebra A is exact if the maximal tensor product by A preserves short exact sequences. The natural algebraic translation of that is that A is flat as a module. So a naive translation of Kirchberg's Embedding Theorem for involutive algebras might be: All flat involutive R -algebras with a countable basis embed $*$ -homomorphically into $L_{2,R}$. However, we will see that this statement is false, at least when $R = \mathbb{Z}$. We also note that in the case where R happens to be a field K this formulation seems very ambitious since *all* algebras over a field are flat. While we do not have a counterexample to the naive translation in this case, it seems unlikely that all algebras with a countable basis should embed into $L_{2,R}$.

The main result of [12] is that every Leavitt path algebra $L_R(E)$ of a countable graph E over a commutative ring R with unit embeds into the Leavitt algebra $L_{2,R}$. The condition of a countable graph ensures the existence of a countable basis, and all Leavitt path algebras are flat as R -modules. (A concrete basis is constructed in [6] for row-finite E , and this result is generalised to all graphs and all rings of coefficients in [4, Corollary 1.5.14].) In the present paper we move slightly outside of the realm of Leavitt path algebras, looking instead at tensor products of Leavitt path algebras. Through the work of [7], we know that for K a field the tensor

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product $L_{2,K} \otimes L_{2,K}$ is not isomorphic to any Leavitt path algebra. A potential embedding of $L_{2,K} \otimes L_{2,K}$ into $L_{2,K}$ is thus a natural question. Indeed, this question is an open problem in the subject, and has attracted enough attention to be included on the Graph Algebra Problem Page,¹ which is a repository of open problems for graph C^* -algebras and Leavitt path algebras that was established after the workshop “Graph Algebras: Bridges between graph C^* -algebras and Leavitt path algebras” held at the Banff International Research Station in April, 2013. This problem is also discussed in [1].

In our main result we provide an answer to this problem in the case $R = \mathbb{Z}$: there is no embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,\mathbb{Z}}$. More precisely, we prove that there is no such unital $*$ -algebra embedding. We restrict our attention to unital embeddings, even when considering $*$ -homomorphisms over a ring R , because it is known that for K a field any homomorphism from a unital ring into $L_{2,K}$ can be “twisted²” into a unital homomorphism because all nonzero idempotents in $L_{2,K}$ are equivalent (see [8, Theorem 3.5]). We discuss the idea of twisting $*$ -homomorphisms into $L_{2,\mathbb{Z}}$ in Section 6. Since $L_{2,\mathbb{Z}}$ is flat (as a \mathbb{Z} -module) and has a countable basis, a consequence of our nonembedding result is that the naive algebraic analogue to Kirchberg’s Embedding Theorem described earlier does not hold.

We prove our main result by studying the unitaries in $L_{2,\mathbb{Z}}$; in particular we use that Thompson’s group V sits naturally as a subgroup \mathcal{U}_V of the unitary group of $L_{2,R}$ for any R . In our second result we eliminate the existence of any unital $*$ -algebra embedding of $L_{2,R} \otimes L_{2,R}$ into $L_{2,R}$ under which the tensor $u \otimes v$ of any two full spectrum unitaries $u, v \in L_{2,R}$ is in \mathcal{U}_V . For a ring of characteristic 0 we can conclude that any potential unital $*$ -algebra embedding of $L_{2,R} \otimes L_{2,R}$ into $L_{2,R}$ must send the tensor of full spectrum unitaries to unitaries in $L_{2,R}$ with nontrivial coefficients. We hope these results and their proofs will inform future work on solving “does $L_{2,R} \otimes L_{2,R}$ embed into $L_{2,R}$ ” for rings other than \mathbb{Z} .

2. PRELIMINARIES ON $L_{2,R}$ AND THOMPSON’S GROUP V

In this section we recall the definition of $L_{2,R}$, and we discuss a representation of $L_{2,R}$ by endomorphisms on the free R -module generated by the infinite paths in the graph underlying $L_{2,R}$. We then discuss the unitaries in $L_{2,R}$ and their relationship to Thompson’s group V .

Standing Assumption. Throughout this paper R will always be a commutative ring with unit.

2.1. The algebra $L_{2,R}$ and a representation. We will be focussed on the graph

$$(2.1) \quad \begin{array}{c} \text{a} \quad \text{b} \\ \text{---} \circ \text{---} \end{array}$$

and its Leavitt path algebra $L_{2,R}$, which is the universal R -algebra generated by elements a, a^*, b, b^* satisfying

$$a^*a = b^*b = 1 = aa^* + bb^*.$$

¹<http://www.math.uh.edu/~tomforde/GraphAlgebraProblems/GraphAlgebraProblemPage.html>

²Formally: Let A be a unital R -algebra and let $\phi: A \rightarrow L_{2,R}$ be a ring homomorphism. Since $\phi(1)$ is an idempotent, there exist $x, y \in L_{2,R}$ such that $xy = \phi(1)$, $yx = 1$. Then $\psi(a) = y\phi(a)x$ defines a unital ring homomorphism from A to $L_{2,R}$. If ϕ is injective so is ψ .

Note that the relations imply that $a^*b = 0 = b^*a$. When R is a field K , $L_{2,K}$ is the Leavitt algebra of module type $(1, 2)$ (see [16]). For every word $\alpha = \alpha_1 \dots \alpha_n$ in a and b we set $\alpha^* := \alpha_n^* \dots \alpha_1^*$. Every element of $L_{2,R}$ is a finite R -linear combination of elements of the form $\alpha\beta^*$, where α and β are words in a and b . The map $\alpha\beta^* \mapsto \beta\alpha^*$ extends to an R -linear involution of $L_{2,R}$. We write $L_{2,R} \otimes L_{2,R}$ to mean the tensor product balanced over R . The tensor product $L_{2,R} \otimes L_{2,R}$ is an involutive R -algebra with $(x \otimes y)^* = x^* \otimes y^*$.

The infinite paths in the graph underlying $L_{2,R}$ is the set $\{a, b\}^{\mathbb{N}}$. We denote the set of finite paths by $\{a, b\}^*$, and by $|\alpha|$ the number of edges, or length, of a path $\alpha \in \{a, b\}^*$. For $\alpha \in \{a, b\}^*$ and $\xi \in \{a, b\}^{\mathbb{N}}$, $\alpha\xi \in \{a, b\}^{\mathbb{N}}$ denotes the obvious concatenation. For each $\alpha \in \{a, b\}^*$ we denote by $Z(\alpha)$ the cylinder set

$$Z(\alpha) = \{\alpha\xi : \xi \in \{a, b\}^{\mathbb{N}}\} \subset \{a, b\}^{\mathbb{N}}.$$

We denote by M the uncountably generated free R -module generated by $\{a, b\}^{\mathbb{N}}$, and by $\text{End}_R(M)$ the endomorphism ring of M . Consider the maps $T_a, T_b : M \rightarrow M$ given by

$$T_a(\xi) = a\xi \quad \text{and} \quad T_b(\xi) = b\xi$$

for all $\xi \in \{a, b\}^{\mathbb{N}}$, and extended to R -linear combinations in the natural way. Also consider $T_a^*, T_b^* : M \rightarrow M$ given by

$$T_a^*(\xi) = \begin{cases} \xi' & \text{if } \xi = a\xi' \text{ for some } \xi' \in \{a, b\}^{\mathbb{N}} \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_b^*(\xi) = \begin{cases} \xi' & \text{if } \xi = b\xi' \text{ for some } \xi' \in \{a, b\}^{\mathbb{N}} \\ 0 & \text{otherwise} \end{cases}$$

for all $\xi \in \{a, b\}^{\mathbb{N}}$, and extended to R -linear combinations in the natural way. Then $T_a, T_a^*, T_b, T_b^* \in \text{End}_R(M)$, and it is straightforward to check that

$$T_a^*T_a = T_b^*T_b = 1 = T_aT_a^* + T_bT_b^*.$$

The universal property of $L_{2,R}$ gives us a unital homomorphism $\pi_T : L_{2,R} \rightarrow \text{End}_R(M)$ satisfying $\pi_T(a) = T_a$, $\pi_T(a^*) = T_a^*$, $\pi_T(b) = T_b$ and $\pi_T(b^*) = T_b^*$. It follows from the Cuntz-Krieger Uniqueness Theorem [21, Theorem 6.5] that π_T is injective.

2.2. Thompson's group V and unitaries in $L_{2,R}$. We are concerned with unitaries in $L_{2,R}$, which are elements u satisfying $u^*u = uu^* = 1$. There are two canonical subsets of the unitary group $\mathcal{U}(L_{2,R})$ of $L_{2,R}$. We denote by \mathcal{U}_1 the subset of unitaries that can be written without coefficients:

$$\mathcal{U}_1 = \left\{ u \in \mathcal{U}(L_{2,R}) : u = \sum_{i=1}^n \alpha_i \beta_i^* \text{ for some distinct pairs } \alpha_i, \beta_i \in \{a, b\}^* \right\}.$$

The other subset, in fact a subgroup, is a homomorphic image of Thompson's group V . Thompson introduced V , as well as the groups F and T , in unpublished notes in 1965. A account of his work can be found in [15]. Thompson's group V can be realised as a collection of homeomorphisms of the Cantor set. Following [18] we think of elements of Thompson's group V as tables

$$\begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}$$

where $\alpha_i, \beta_i \in \{a, b\}^*$ such that

$$\{a, b\}^{\mathbb{N}} = \bigsqcup_{i=1}^n Z(\alpha_i) = \bigsqcup_{i=1}^n Z(\beta_i).$$

Each table determines a homeomorphism of $\{a, b\}^{\mathbb{N}}$ which maps an infinite path of the form $\beta_i \xi$ to $\alpha_i \xi$. The set of homeomorphisms defined by tables is a group under composition. The calculation that gives the product of two tables as another table mirrors exactly the one which gives the product of two spanning elements $\sum_{i=1}^n \alpha_i \beta_i^*$ as another spanning element; that is, which shows that \mathcal{U}_1 is a subgroup. So the map

$$\begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \mapsto \sum_{i=1}^n \alpha_i \beta_i^*$$

is a group homomorphism of V into $\mathcal{U}(L_{2,R})$. We denote its image by \mathcal{U}_V . By definition of \mathcal{U}_1 and \mathcal{U}_V we have $\mathcal{U}_V \subseteq \mathcal{U}_1$. We now show that the reverse containment holds when R has characteristic 0. The case for fields is covered by [19, Lemma 3.3].

Lemma 2.1. *If R has characteristic 0, then $\mathcal{U}_V = \mathcal{U}_1$.*

Proof. We need to show that if $u = \sum_{i=1}^n \alpha_i \beta_i^*$ is unitary, then

$$\{a, b\}^{\mathbb{N}} = \bigsqcup_{i=1}^n Z(\alpha_i) = \bigsqcup_{i=1}^n Z(\beta_i).$$

We work with the representation $\pi_T: L_{2,R} \rightarrow \text{End}_R(M)$. We know that $\pi_T(u^*u) = \pi_T(uu^*)$ is the identity endomorphism of M . First suppose $\bigcup_{i=1}^n Z(\beta_i) \neq \{a, b\}^{\mathbb{N}}$. Then for all $\xi \in \{a, b\}^{\mathbb{N}} \setminus (\bigcup_{i=1}^n Z(\beta_i))$ we have $T_{\beta_i}^* \xi = 0$, and hence

$$\pi_T(u^*u)\xi = \pi_T(u^*)\left(\sum_{i=1}^n T_{\alpha_i}(T_{\beta_i}^* \xi)\right) = 0,$$

which contradicts that $\pi_T(u^*u)$ is the identity endomorphism on M . So we have $\bigcup_{i=1}^n Z(\beta_i) = \{a, b\}^{\mathbb{N}}$. A similar argument using $\pi_T(uu^*)$ shows that $\bigcup_{i=1}^n Z(\alpha_i) = \{a, b\}^{\mathbb{N}}$.

We now claim that $\{Z(\beta_i) : 1 \leq i \leq n\}$ are mutually disjoint. Let $m = \max\{|\beta_i| : 1 \leq i \leq n\}$, and for each $1 \leq i \leq n$ we let X_i denote the set of paths of length $m - |\beta_i|$. Since each $\sum_{\gamma \in X_i} \gamma \gamma^* = 1$, we have

$$u = \sum_{i=1}^n \alpha_i \beta_i^* = \sum_{i=1}^n \alpha_i \left(\sum_{\gamma \in X_i} \gamma \gamma^* \right) \beta_i^* = \sum_{i=1}^n \sum_{\gamma \in X_i} \alpha_i \gamma (\beta_i \gamma)^*.$$

We relabel this sum and write $u = \sum_{i=1}^p \mu_i \nu_i^*$, where each ν_i has the same length. We will prove that each ν_i is distinct. Suppose not for contradiction. By relabelling we can assume without loss of generality that $\nu_1 = \nu_2 = \cdots = \nu_j$ for some $j \geq 2$, and $\nu_1 \neq \nu_k$ for all $j+1 \leq k \leq p$. Then for any $\xi \in \{a, b\}^{\mathbb{N}}$ we have

$$(2.2) \quad \pi_T(u^*u)\nu_1 \xi = \pi_T(u^*)\left(\sum_{i=1}^j \mu_i \xi\right) = \sum_{k=1}^p \sum_{i=1}^j T_{\nu_k} T_{\mu_k}^* \mu_i \xi = j\nu_1 \xi + x,$$

where $x \in M$ is a linear combination with positive integer coefficients of basis elements of the form $\nu_k \eta$ for some $1 \leq k \leq p$ and $\eta \in \{a, b\}^{\mathbb{N}}$. Since the characteristic of R is 0, it follows that the expression on the right of (2.2) is not $\nu_1 \xi$. But this contradicts that $\pi_T(u^*u)$ is the identity endomorphism. Hence each ν_i is distinct.

Now, each ν_j has the form $\beta_i \gamma$. So if each ν_j is distinct, and because they are all of the same length, then it follows that no β_i can be the subword of any other β_k for $1 \leq i, k \leq n$. Hence $\{Z(\beta_i) : 1 \leq i \leq n\}$ are mutually disjoint, as claimed. A similar argument using $\pi_T(uu^*)$ shows that the $Z(\alpha_i)$ must be mutually disjoint. \square

The following example shows that the assumption of characteristic 0 is necessary.

Example 2.2. Consider the field $K = \mathbb{Z}/2\mathbb{Z}$. In $M_4(\mathbb{Z}/2\mathbb{Z})$ the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

is a self-adjoint unitary. We can then use an embedding $M_4(\mathbb{Z}/2\mathbb{Z}) \hookrightarrow L_{2,\mathbb{Z}/2\mathbb{Z}}$ (see [12, Example 5.3]) to get a unitary $u \in L_{2,\mathbb{Z}/2\mathbb{Z}}$ given by

$$u = (ab + ba + bb)a^*a^* + (aa + ba + bb)b^*a^* + (aa + ab + bb)a^*b^* + (aa + ab + ba)b^*b^*.$$

Clearly $u \in \mathcal{U}_1$. We claim that $u \notin \mathcal{U}_V$. When we represent the elements of $L_{2,\mathbb{Z}/2\mathbb{Z}}$ as endomorphisms of the free module with basis $\{a, b\}^{\mathbb{N}}$, as discussed in Section 2.1, the elements of \mathcal{U}_V will map basis elements to basis elements, but for any $\xi \in \{a, b\}^{\mathbb{N}}$ we see that

$$uaa\xi = ab\xi + ba\xi + bb\xi.$$

Hence $u \notin \mathcal{U}_V$.

3. EMBEDDING THE LAURENT POLYNOMIALS $L_R[w, w^{-1}, z, z^{-1}]$

We prove a positive and negative embedding result for the Laurent polynomials $L_R[w, w^{-1}, z, z^{-1}]$ in two commuting variables over R . We first show that $L_R[w, w^{-1}, z, z^{-1}]$ embeds into $L_{2,R} \otimes L_{2,R}$. We then show that there is no $*$ -algebraic embedding of $L_R[w, w^{-1}, z, z^{-1}]$ into $L_{2,R}$ mapping the unitaries w and z into the image $\mathcal{U}_V \subset \mathcal{U}(L_{2,R})$ of Thompson's group V .

Proposition 3.1. *Let R be a commutative ring with unit. Then $L_R[w, w^{-1}, z, z^{-1}]$ embeds unitaly into $L_{2,R} \otimes L_{2,R}$ as $*$ -algebras.*

Remark 3.2. To prove Proposition 3.1 we use the following general observation about flat modules. If \mathcal{E} is a flat right R -module, and $\phi: \mathcal{M} \rightarrow \mathcal{N}$ is an embedding of left R -modules, then from the exact sequence $0 \rightarrow \mathcal{M} \hookrightarrow \mathcal{N}$, we get the exact sequence

$$0 \rightarrow \mathcal{E} \otimes \mathcal{M} \xrightarrow{\text{id} \otimes \phi} \mathcal{E} \otimes \mathcal{N}.$$

Hence $\mathcal{E} \otimes \mathcal{M}$ embeds into $\mathcal{E} \otimes \mathcal{N}$. Now if \mathcal{E} and \mathcal{M} are flat R -modules (so both left and right R -modules), and \mathcal{E} embeds into \mathcal{M} , then $\mathcal{E} \otimes \mathcal{E}$ embeds into $\mathcal{E} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \mathcal{E}$ embeds into $\mathcal{M} \otimes \mathcal{M}$. Using the flip isomorphism then gives an embedding

$$\mathcal{E} \otimes \mathcal{E} \hookrightarrow \mathcal{E} \otimes \mathcal{M} \xrightarrow{\cong} \mathcal{M} \otimes \mathcal{E} \hookrightarrow \mathcal{M} \otimes \mathcal{M}.$$

Composing with a flip isomorphism once more, we see that if $\psi: \mathcal{E} \rightarrow \mathcal{M}$ is an embedding, then $\psi \otimes \psi$ is an embedding of $\mathcal{E} \otimes \mathcal{E}$ into $\mathcal{M} \otimes \mathcal{M}$.

Proof of Proposition 3.1. We have $L_R[w, w^{-1}, z, z^{-1}] \cong L_R[w, w^{-1}] \otimes L_R[z, z^{-1}]$, and it follows from [12, Theorem 4.1] that $L_R[w, w^{-1}]$ embeds unitaly into $L_{2,R}$ as $*$ -algebras. Moreover, $L_R[w, w^{-1}]$ and $L_{2,R}$ have countable bases and hence are flat. So the result follows from Remark 3.2 applied to $\mathcal{E} = L_R[w, w^{-1}]$ and $\mathcal{M} = L_{2,R}$. \square

We state our nonembedding result.

Theorem 3.3. *Let R be a commutative ring with unit. There does not exist a $*$ -algebraic embedding*

$$\phi: L_R[w, w^{-1}, z, z^{-1}] \rightarrow L_{2,R}$$

with $\phi(w), \phi(z) \in \mathcal{U}_V$.

The result is a consequence of the structure of Thompson's group V . The following proposition extracts results about V from [11] and reformulates them in to our setting.

Proposition 3.4. *If $u, v \in \mathcal{U}_V$ commute, then there exists a nonzero polynomial $q(w, z) \in R[w, z]$ such that $q(u, v) = 0$.*

In the proof we will adopt the notational conventions of [11]. However to fit with the notation of the rest of this paper, we will choose $\{a, b\}^{\mathbb{N}}$ as our model for the Cantor set rather than $\{0, 1\}^{\mathbb{N}}$.

Proof of Proposition 3.4. Fix commuting $u, v \in \mathcal{U}_V$. Since \mathcal{U}_V is the homomorphic image of V and since V is simple, there exist commuting elements $g, h \in V$ that are mapped to u, v . By [14, Proposition 10.1] there exist $m, n \in \mathbb{N}$ such that all finite orbits of g have length at most m and all finite orbits of h have length at most n . Let $k = 1 \cdot 2 \cdot \dots \cdot m$ and $l = 1 \cdot 2 \cdot \dots \cdot n$, then g^k and h^l have no nontrivial finite orbits. If $q_0(w, z) \in R[w, z]$ is such that $q_0(u^k, v^l) = 0$, then $q(w, z) = q_0(w^k, z^l)$ satisfies $q(u, v) = 0$. So it suffices to prove the result under the assumption that g and h have no nontrivial finite orbits. From here on out we will assume that.

We wish to use [11, Lemmas 2.5 and 2.6] to construct our polynomial. The notion of a flow graph and the components of support of an element of Thompson's group V is introduced in [11, Section 2.4]. Denote the components of support of g by X_1, X_2, \dots, X_m and the components of support of h by Y_1, Y_2, \dots, Y_n .

From the description of components of support we see that the X_i are disjoint, that each X_i has the form $X_i = \bigsqcup_{j=1}^{m_i} Z(\alpha_{i,j})$ for $m_i \in \mathbb{N}$ and paths $\alpha_{i,j} \in \{a, b\}^*$, and that g acts as the identity on $X_0 = \{a, b\}^{\mathbb{N}} \setminus \bigcup_{i=1}^m X_i$. For $i = 1, 2, \dots, m$ we define projections $r_i \in L_{2,R}$ by

$$r_i = \sum_{j=1}^{m_i} \alpha_{i,j} \alpha_{i,j}^*.$$

By construction the r_i are orthogonal, and since g maps X_i to itself, we have $r_i u r_i = u r_i$. We define a projection $r_0 \in L_{2,R}$ by

$$r_0 = 1 - \sum_{i=1}^m r_i.$$

Then $r_0 u r_0 = u r_0 = r_0$ since g acts as the identity outside of its components of support. We have

$$u = \sum_{i=0}^m r_i u r_i.$$

Similarly we associate orthogonal projections $s_1, \dots, s_n \in L_{2,R}$ to the components of support of h and define

$$s_0 = 1 - \sum_{i=1}^n s_i$$

Then

$$v = \sum_{i=0}^n s_i v s_i.$$

Since g and h commute [11, Lemma 2.5(3)] tells us that the components of support of g, h are either disjoint or equal. Thus for $j = 1, 2, \dots, n$, s_j either equals r_i for some i , or s_j is a subprojection of r_0 . Similarly r_i is a subprojection of s_0 if it is not equal to one of the s_j .

Let $I = \{1 \leq i \leq m : r_i = s_j \text{ for some } 1 \leq j \leq n\}$, that is I are the indices of the common components of support. Let l be the size of I . We let p_1, p_2, \dots, p_l be the associated projections and we put $p_0 = r_0$. Then

$$1 - \sum_{k=0}^l p_k = \sum_{i \notin I \cup \{0\}} r_i$$

is a subprojection of s_0 . Let $p_{l+1} = 1 - \sum_{k=0}^l p_k$.

We now have orthogonal projections p_0, p_1, \dots, p_{l+1} such that

- $\sum_{k=0}^{l+1} p_k = 1$,
- for $k = 1, 2, \dots, l$ we have $p_k = r_{i(k)} = s_{j(k)}$ for some $1 \leq i(k) \leq m$, $1 \leq j(k) \leq n$,
- $p_0 = r_0$,
- $p_{l+1} \leq s_0$,

Furthermore, since p_{l+1} is a sum of projections onto components of support of g , we also have that $p_{l+1}up_{l+1} = up_{l+1}$. As p_{l+1} is a subprojection of s_0 , and so corresponds to a subset where h acts as the identity, we also have $p_{l+1}vp_{l+1} = vp_{l+1} = p_{l+1}$. Since p_0 will be a (possibly empty) sum of projections onto components of support of h and a (possibly zero) subprojection of s_0 we see that $p_0vp_0 = vp_0$. Therefore we also have

- $u = \sum_{k=0}^{l+1} p_k up_k$, and
- $v = \sum_{k=0}^{l+1} p_k vp_k$.

For each $1 \leq k \leq l$ we can use [11, Lemma 2.6] to find $m_k, n_k \in \mathbb{N}$ such that

$$(p_k up_k)^{m_k} = (p_k vp_k)^{n_k}.$$

We define a polynomial $q \in R[z, w]$ by

$$q(w, z) = (wz - z) \left(\prod_{k=1}^l (w^{m_k} - z^{n_k}) \right) (z - wz).$$

Note that q is nonzero since the coefficient of the only $w^{1+m_1+m_2+\dots+m_l+1}z^2$ term is -1 .

For $1 \leq k \leq l$ we see that $q(p_k up_k, p_k vp_k) = 0$ because one of the middle factors of q will be zero. By the definition of p_0 we have $p_0 up_0 = p_0$ so $p_0 up_0 p_0 vp_0 = p_0 vp_0$ and therefore $q(p_0 up_0, p_0 vp_0) = 0$ since the left most factor of q will be zero. Using that $p_{l+1}vp_{l+1} = p_{l+1}$ and the right most factor of q similar reasoning shows that $q(p_{l+1}up_{l+1}, p_{l+1}vp_{l+1}) = 0$.

Because the p_k are orthogonal we see that

$$q(u, v) = q\left(\sum_{k=0}^l p_k up_k, \sum_{k=0}^l p_k vp_k\right) = \sum_{k=0}^l q(p_k up_k, p_k vp_k) = 0. \quad \square$$

Proof of Theorem 3.3. Suppose $\phi: L_R[w, w^{-1}, z, z^{-1}] \rightarrow L_{2,R}$ is a $*$ -homomorphism with $\phi(w), \phi(z) \in \mathcal{U}_V$. Then we know from Proposition 3.4 that there is a polynomial $q(w, z) \in R[w, z]$ with $q(\phi(w), \phi(z)) = 0$. So $\phi(q(w, z)) = q(\phi(w), \phi(z)) = 0$, and hence ϕ is not injective. \square

4. $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ DOES NOT EMBED IN $L_{2,\mathbb{Z}}$

We now state our main result.

Theorem 4.1. *There is no unital $*$ -algebraic embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,\mathbb{Z}}$.*

To prove Theorem 4.1 we need a number of results. We have decided to stay only in the generality of the integers throughout, although it is probable that some results can be generalised to cater for more general rings. The first step is to give a characterisation of $\mathcal{U}(L_{2,\mathbb{Z}})$, which we do in Proposition 4.4.

Definition 4.2. An expression

$$\sum_{i=1}^n \lambda_i \alpha_i \beta_i^*$$

is in *reduced form* if each $\lambda_i \in \mathbb{Z} \setminus \{0\}$ and $\bigsqcup_{i=1}^n Z(\alpha_i) = \{a, b\}^{\mathbb{N}} = \bigsqcup_{i=1}^n Z(\beta_i)$.

Lemma 4.3. Let $u \in \mathcal{U}(L_{2,\mathbb{Z}})$ be given and suppose

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

with the pairs (α_i, β_i) distinct and $\lambda_i \in \mathbb{Z} \setminus \{0\}$. If all the β_i have the same length then they are all distinct.

Proof. We work with the representation $\pi_T: L_{2,\mathbb{Z}} \rightarrow \text{End}_{\mathbb{Z}}(M)$ discussed in Section 2.1.

Suppose, for contradiction, that some of the β_i are equal. By relabelling we can assume without loss of generality that $\beta_1 = \beta_2 = \dots = \beta_j$, and $\beta_1 \neq \beta_k$ for all $j+1 \leq k \leq n$ for some $j \geq 2$. Let ξ be the aperiodic path $abaabbbaabbb \dots \in \{a, b\}^{\mathbb{N}}$. Then we have

$$(4.1) \quad \pi_T(u^*u)\beta_1\xi = \pi_T(u^*)\left(\sum_{i=1}^j \lambda_i \alpha_i \xi\right) = \sum_{k=1}^n \sum_{i=1}^j \lambda_i \lambda_k T_{\beta_k} T_{\alpha_k}^* \alpha_i \xi.$$

Now, for every pair $1 \leq i \neq k \leq j$, if α_i extends α_k we write $\gamma_{i,k}$ for the path with $\alpha_i = \alpha_k \gamma_{i,k}$, and if ξ extends $\gamma_{i,k}$, we write $\xi_{i,k}$ for the infinite path with $\xi = \gamma_{i,k} \xi_{i,k}$. Then the right-hand expression in (4.1) is given by

$$(4.2) \quad \left(\sum_{i=1}^j \lambda_i^2\right) \beta_1 \xi + \sum_{\substack{1 \leq i \neq k \leq j \\ \alpha_i = \alpha_k \gamma_{i,k}}} \lambda_i \lambda_k \beta_1 \gamma_{i,k} \xi + \sum_{\substack{1 \leq i \neq k \leq j \\ \xi = \gamma_{i,k} \xi_{i,k}}} \lambda_i \lambda_k \beta_1 \xi_{i,k} + m,$$

where $m \in M$ is a linear combination of basis elements of the form $\beta_k \eta$ for some $j+1 \leq k \leq n$ and $\eta \in \{a, b\}^{\mathbb{N}}$. By the choice ξ we have $\beta_1 \xi \neq \beta_1 \gamma_{i,k} \xi$ and $\beta_1 \xi \neq \beta_1 \xi_{i,k}$ for any $1 \leq i \neq k \leq j$. Since $\pi_T(uu^*)$ is the identity on M , we must therefore have that

$$\sum_{\substack{1 \leq i \neq k \leq j \\ \alpha_i = \alpha_k \gamma_{i,k}}} \lambda_i \lambda_k \beta_1 \gamma_{i,k} \xi + \sum_{\substack{1 \leq i \neq k \leq j \\ \xi = \gamma_{i,k} \xi_{i,k}}} \lambda_i \lambda_k \beta_1 \xi_{i,k} + m = 0$$

and

$$\left(\sum_{i=1}^j \lambda_i^2\right) \beta_1 \xi = \beta_1 \xi.$$

Hence $\sum_{i=1}^j \lambda_i^2 = 1$ which contradicts that $j > 1$. So all the β_i are distinct. \square

Proposition 4.4. Every unitary $u \in \mathcal{U}(L_{2,\mathbb{Z}})$ can be written in reduced form. Furthermore, if

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

is in reduced form then $\lambda_i \in \{-1, 1\}$ for all $i = 1, 2, \dots, n$.

Proof. Let $u \in \mathcal{U}(L_{2,\mathbb{Z}})$ be given and write

$$u = \sum_{i=1}^m \lambda'_i \alpha'_i \beta_i'^*.$$

As argued in the proof of Lemma 2.1 we can extend the paths α'_i, β'_i to write u like

$$(4.3) \quad u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*$$

where the β_i all have equal length. By simply collecting like terms and discarding terms where $\lambda_i = 0$, we may assume that the pairs (α_i, β_i) are distinct and that $\lambda_i \in \mathbb{Z} \setminus \{0\}$. It now follows from Lemma 4.3 that all the β_i are distinct and hence they have disjoint cylinder sets.

We claim that the α_i also have disjoint cylinder sets. To see this, consider the unitary

$$u^* = \sum_{i=1}^n \lambda_i \beta_i \alpha_i^*.$$

Similar to in the proof of Lemma 2.1 we let $m = \max\{|\alpha_i| : 1 \leq i \leq n\}$, and for each $1 \leq i \leq n$ we let X_i denote the set of paths of length $m - |\alpha_i|$. Since each $\sum_{\gamma \in X_i} \gamma \gamma^* = 1$, we have

$$u^* = \sum_{i=1}^n \lambda_i \beta_i \alpha_i^* = \sum_{i=1}^n \lambda_i \beta_i \left(\sum_{\gamma \in X_i} \gamma \gamma^* \right) \alpha_i^* = \sum_{i=1}^n \sum_{\gamma \in X_i} \lambda_i \beta_i \gamma (\alpha_i \gamma)^*.$$

For any $\gamma, \gamma' \in \{a, b\}^*$ we have that $\beta_k \gamma = \beta_j \gamma'$ if and only if $k = j$ and $\gamma = \gamma'$, since the β_i all have the same length. So it follows from Lemma 4.3 that the $\alpha_i \gamma$ are distinct. Hence no α_k is an initial segment of any other α_j , and the cylinder sets $Z(\alpha_i)$ are disjoint.

It is easy to see, as in the proof of Lemma 2.1, that the union of the cylinder sets of both the β_i and the α_i must equal $\{a, b\}^{\mathbb{N}}$, so we have

$$\bigsqcup_{i=1}^n Z(\alpha_i) = \{a, b\}^{\mathbb{N}} = \bigsqcup_{i=1}^n Z(\beta_i).$$

Thus (4.3) is a reduced form of u .

Finally, if

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

is a reduced form of u , then for any $\xi \in \{a, b\}^{\mathbb{N}}$ and any $1 \leq k \leq n$ we have

$$\beta_k \xi = \pi_T(u^* u) \beta_k \xi = \lambda_k^2 \beta_k \xi,$$

so $\lambda_k^2 = 1$. Therefore $\lambda_i \in \{-1, 1\}$ for all $1 \leq i \leq n$. \square

Notation 4.5. For a unitary u with reduced form $u = \sum_{i=1}^k \lambda_i \alpha_i \beta_i^* \in \mathcal{U}(L_{2,\mathbb{Z}})$ we denote

$$u_+ = \sum_{i=1}^k \alpha_i \beta_i^* \in L_{2,\mathbb{Z}}.$$

Proposition 4.6. *Unitaries in $L_{2,\mathbb{Z}}$ have the following properties.*

- (1) If $u \in \mathcal{U}(L_{2,\mathbb{Z}})$, then $u_+ \in \mathcal{U}_V$.
- (2) If $u, v \in \mathcal{U}(L_{2,\mathbb{Z}})$ commute, then u_+ and v_+ commute.
- (3) If $u, v \in \mathcal{U}(L_{2,\mathbb{Z}})$ commute, and $0 \neq q(w, z) \in \mathbb{Z}[w, z]$ with $q(u_+, v_+) = 0$, then there exists $0 \neq \tilde{q}(w, z) \in \mathbb{Z}[w, z]$ with $\tilde{q}(u, v) = 0$.

To prove Proposition 4.6 we use the following results about endomorphisms of free abelian groups, i.e. \mathbb{Z} -modules. For f, g endomorphisms of a free abelian group G we denote by $\text{supp}(f) = \{m \in G : f(m) \neq 0\}$, and by $[f, g]$ the commutator $fg - gf$.

Lemma 4.7. *Let G be a free abelian group with basis \mathcal{B} . Let $f, g, s, t \in \text{End}(G)$ satisfy*

- (i) *the image of a basis element is another basis element or zero; and*
- (ii) $G = \text{supp}(f) \sqcup \text{supp}(g) \sqcup \{0\} = \text{supp}(s) \sqcup \text{supp}(t) \sqcup \{0\}$.

Then

$$[f - g, s - t] = 0 \implies [f + g, s + t] = 0.$$

Proof. We have

$$\begin{aligned} [f - g, s - t] = 0 &\iff [f, s] - [f, t] - [g, s] + [g, t] = 0 \\ &\iff [f, s] + [g, t] = [f, t] + [g, s], \end{aligned}$$

and

$$\begin{aligned} [f + g, s + t] = 0 &\iff [f, s] + [f, t] + [g, s] + [g, t] = 0 \\ &\iff [f, s] + [g, t] = -([f, t] + [g, s]). \end{aligned}$$

So we need to prove that

$$(4.4) \quad [f, s] + [g, t] = [f, t] + [g, s] \implies [f, s] + [g, t] = 0.$$

From here on out we will assume the left hand side (4.4).

It suffices to check (4.4) on \mathcal{B} . To do this we partition \mathcal{B} into four sets:

$$\begin{aligned} \mathcal{B}_{f,s} &= \mathcal{B} \cap \text{supp}(f) \cap \text{supp}(s), & \mathcal{B}_{f,t} &= \mathcal{B} \cap \text{supp}(f) \cap \text{supp}(t), \\ \mathcal{B}_{g,s} &= \mathcal{B} \cap \text{supp}(g) \cap \text{supp}(s), & \mathcal{B}_{g,t} &= \mathcal{B} \cap \text{supp}(g) \cap \text{supp}(t). \end{aligned}$$

For $\xi \in \mathcal{B}_{f,s}$ we have that $[g, t]\xi = 0$ and

$$([f, t] + [g, s])\xi = ft\xi - tf\xi + gs\xi - sg\xi = -tf\xi + gs\xi.$$

Hence, on $\mathcal{B}_{f,s}$ the left hand side of (4.4) reduces to

$$(4.5) \quad fs - sf = -tf + gs.$$

Reordering the terms we get

$$(4.6) \quad (f - g)s = (s - t)f.$$

Fix $\xi \in \mathcal{B}_{f,s}$. As f, g, s, t all map basis elements to basis elements or zero, we cannot have $s\xi \in \text{supp}(g)$ and $f\xi \in \text{supp}(s)$, since then we would get the nonzero basis element $sf\xi$ equal to the negative of the nonzero basis element $gs\xi$. Similarly, we cannot have $s\xi \in \text{supp}(f)$ and $f\xi \in \text{supp}(t)$. Therefore, if $s\xi \in \text{supp}(f)$ then we must have $f\xi \in \text{supp}(s)$, and if $s\xi \in \text{supp}(g)$ then we must have $f\xi \in \text{supp}(t)$. In the first case, we see directly from (4.5) that $[f, s]\xi = (fs - sf)\xi = 0$. In the other case (4.6) yields that $-gs\xi = -tf\xi$, which by (4.5) gives that $[f, s]\xi = 0$. As noted earlier $[g, t]\xi = 0$ for all $\xi \in \mathcal{B}_{f,s}$, so we have that $[f, s] + [g, t] = 0$ on $\mathcal{B}_{f,s}$. Hence (4.4) is satisfied on $\mathcal{B}_{f,s}$. We check the remaining three cases in a similar fashion:

For $\xi \in \mathcal{B}_{f,t}$ we have that $[g, s]\xi = 0$ and

$$([f, s] + [g, t])\xi = -sf\xi + gt\xi.$$

So the left hand side of (4.4) reduces to

$$-sf + gt = ft - tf,$$

on $\mathcal{B}_{f,t}$. Reordering we get

$$(t - s)f = (f - g)t.$$

Arguing as above, we note that for $\xi \in \mathcal{B}_{f,t}$ we either have $f\xi \in \text{supp}(t)$ and $t\xi \in \text{supp}(f)$ or we have $f\xi \in \text{supp}(s)$ and $t\xi \in \text{supp}(g)$. Also as above, we see that in both those cases we get $[f, t]\xi = 0$. Hence on $\mathcal{B}_{f,t}$ we have

$$0 = [f, t] + [g, s] = [f, s] + [g, t].$$

So (4.4) is satisfied on $\mathcal{B}_{f,t}$.

On $\mathcal{B}_{g,s}$ we have $[f, t] = 0$ and $[f, s] + [g, t] = fs - tg$, so that after reordering the terms, the left hand side of (4.4) reads:

$$(f - g)s = (t - s)g.$$

Similar to the above, we note that for $\xi \in \mathcal{B}_{g,s}$ we either have $s\xi \in \text{supp}(f)$ and $g\xi \in \text{supp}(t)$, or we have $s\xi \in \text{supp}(g)$ and $g\xi \in \text{supp}(s)$. In either case, we deduce that $[g, s] = 0$. Hence $[f, t] + [g, s] = 0$ on $\mathcal{B}_{g,s}$, and so (4.4) holds on $\mathcal{B}_{g,s}$.

Finally, we see that $[f, s] = 0$ and $[f, t] + [g, s] = ft - sg$ on $\mathcal{B}_{g,t}$. By reordering the terms, we can write the left hand side of (4.4) as

$$(s - t)g = (f - g)t.$$

We conclude, as above, that for $\xi \in \mathcal{B}_{g,s}$ we either have $g\xi \in \text{supp}(s)$ and $t\xi \in \text{supp}(f)$, or we have $g\xi \in \text{supp}(t)$ and $t\xi \in \text{supp}(g)$. Either way, we get that $[g, t] = 0$ and therefore that $[f, s] + [g, t] = 0$ on $\mathcal{B}_{g,t}$. Hence (4.4) holds on $\mathcal{B}_{g,t}$. \square

Lemma 4.8. *Let G be a free abelian group with basis \mathcal{B} . Let $f, g, s, t \in \text{End}(G)$ satisfy*

- (i) *the image of a basis vector is another basis vector or zero;*
- (ii) *$G = \text{supp}(f) \sqcup \text{supp}(g) \sqcup \{0\} = \text{supp}(s) \sqcup \text{supp}(t) \sqcup \{0\}$; and*
- (iii) *$f + g, s + t$ and $f - g, s - t$ are pairs of commuting automorphisms.*

If $0 \neq q(w, z) \in \mathbb{Z}[w, z]$ with $q(f + g, s + t) = 0$, then there exists $0 \neq \tilde{q}(w, z) \in \mathbb{Z}[w, z]$ with $\tilde{q}(f - g, s - t) = 0$.

Proof. Fix nonzero $q(w, z) \in \mathbb{Z}[w, z]$ with $q(f + g, s + t) = 0$. Since multiplying $q(w, z)$ by wz does not change that evaluation at $(f + g, s + t)$ is zero, we will assume for convenience that

$$q(w, z) = \sum_{i,j=1}^n k_{i,j} w^i z^j,$$

for some $n \in \mathbb{N}$.

For every $1 \leq i, j \leq n$ let $\Omega^{i,j}$ be the set of words in $\{f, g, s, t\}$ of length $i + j$ and with first i letters in $\{f, g\}$ and last j letters in $\{s, t\}$. Under composition, we view each $\Omega^{i,j}$ as a subset of $\text{End}(G)$. Let \mathcal{C} be the collection of choice functions on $\{\Omega^{i,j} : 1 \leq i, j \leq n\}$. We claim that for every $\xi \in \mathcal{B}$ there is a unique choice function $\varphi_\xi \in \mathcal{C}$ such that $\varphi_\xi(\Omega^{i,j})\xi \neq 0$ for all $1 \leq i, j \leq n$. To see that there is a choice function φ with this property, fix $1 \leq i, j \leq n$. Let

$$\omega_{i+j}^{i,j} \in \{s, t\} \text{ with } \omega_{i+j}^{i,j}\xi \neq 0,$$

$$\omega_l^{i,j} \in \{s, t\} \text{ with } \omega_l^{i,j}(\omega_{l+1}^{i,j} \dots \omega_{i+j}^{i,j}\xi) \neq 0$$

for each $i + 1 \leq l \leq i + j - 1$, and

$$\omega_m^{i,j} \in \{f, g\} \text{ with } \omega_m^{i,j}(\omega_{m+1}^{i,j} \dots \omega_{i+j}^{i,j}\xi) \neq 0$$

for each $1 \leq m \leq i$. Then $\varphi \in \mathcal{C}$ given by $\varphi(\Omega^{i,j}) = \omega_1^{i,j} \dots \omega_{i+j}^{i,j}$ satisfies $\varphi(\Omega^{i,j})\xi \neq 0$ for all $1 \leq i, j \leq n$. We now note that, because the supports of f and g partition $G \setminus \{0\}$ and the supports of s and t partition $G \setminus \{0\}$, every choice of $\omega_1^{i,j}, \dots, \omega_{i+j}^{i,j}$ is unique. Hence $\varphi_\xi = \varphi$ is the unique choice function with the desired property.

For each $\varphi \in \mathcal{C}$ let

$$\mathcal{B}_\varphi = \{\xi \in \mathcal{B} : \varphi(\Omega^{i,j})\xi \neq 0 \text{ for all } 1 \leq i, j \leq n\}.$$

So $\mathcal{B}_\varphi = \{\xi \in \mathcal{B} : \varphi_\xi = \varphi\}$. By the uniqueness of each φ_ξ we have

$$\mathcal{B} = \bigsqcup_{\varphi \in \mathcal{C}} \mathcal{B}_\varphi.$$

Fix $\varphi \in \mathcal{C}$. Then on \mathcal{B}_φ we have

$$q(f + g, s + t) = \sum_{i,j=1}^n k_{i,j}(f + g)^i(s + t)^j = \sum_{i,j=1}^n k_{i,j}\varphi(\Omega^{i,j}),$$

and

$$q(f - g, s - t) = \sum_{i,j=1}^n k_{i,j}(f - g)^i(s - t)^j = \sum_{i,j=1}^n (-1)^{\alpha_{(\varphi,i,j)}} k_{i,j}\varphi(\Omega^{i,j}),$$

where $\alpha_{(\varphi,i,j)}$ is the sum of the number of instances of g and t appearing in the word $\varphi(\Omega^{i,j})$. Let

$$q_\varphi(w, z) = \sum_{i,j=1}^n (-1)^{\alpha_{(\varphi,i,j)}} k_{i,j} w^i z^j \in \mathbb{Z}[w, z].$$

Then on \mathcal{B}_φ we have

$$\begin{aligned} q_\varphi(f - g, s - t) &= \sum_{i,j=1}^n (-1)^{2\alpha_{(\varphi,i,j)}} k_{i,j}\varphi(\Omega^{i,j}) \\ &= \sum_{i,j=1}^n k_{i,j}\varphi(\Omega^{i,j}) \\ &= q(f + g, s + t) \\ &= 0. \end{aligned}$$

Define

$$\tilde{q}(w, z) = \prod_{\varphi \in \mathcal{C}} q_\varphi(w, z) \in \mathbb{Z}[w, w^{-1}, z, z^{-1}].$$

Note that since each $\Omega^{i,j}$ is finite the set \mathcal{C} is finite, so the product is finite. For each $\xi \in \mathcal{B}$ we use that polynomials commute to get

$$\tilde{q}(f - g, s - t)\xi = \left(\prod_{\varphi \neq \varphi_\xi} q_\varphi(f - g, s - t) \right) q_{\varphi_\xi}(f - g, s - t)\xi = 0.$$

Hence $\tilde{q}(f - g, s - t) = 0$. Finally, to see that $\tilde{q}(w, z)$ is not the zero polynomial, let

$$i_0 = \max\{i : k_{i,j} \neq 0\} \quad \text{and} \quad j_0 = \max\{j : k_{i_0,j} \neq 0\}.$$

Then $\tilde{q}(z, w)$ contains the nonzero term

$$(-1)^{\sum_{\varphi \in \mathcal{C}} \alpha_{(\varphi,i_0,j_0)}} k_{i_0,j_0}^{|\mathcal{C}|} (w^{i_0} z^{j_0})^{|\mathcal{C}|}.$$

□

Proof of Proposition 4.6. Property (1) follows immediately from the characterisation of $\mathcal{U}(L_{2,\mathbb{Z}})$ in Proposition 4.4.

For (2) and (3) we let $u = \sum_{i=1}^m \lambda_i \alpha_i \beta_i^*$ and $v = \sum_{j=1}^n \mu_j \gamma_j \delta_j^*$ be commuting unitaries. Let

$$\begin{aligned} u_{+,+} &= \sum_{\substack{1 \leq i \leq m \\ \lambda_i = 1}} \alpha_i \beta_i^*, & u_{+,-} &= \sum_{\substack{1 \leq i \leq m \\ \lambda_i = -1}} \alpha_i \beta_i^*, \\ v_{+,+} &= \sum_{\substack{1 \leq j \leq n \\ \mu_j = 1}} \gamma_j \delta_j^*, & v_{+,-} &= \sum_{\substack{1 \leq j \leq n \\ \mu_j = -1}} \gamma_j \delta_j^*. \end{aligned}$$

So we have

$$\begin{aligned} u &= u_{+,+} - u_{+,-}, & u_+ &= u_{+,+} + u_{+,-}, \\ v &= v_{+,+} - v_{+,-}, & v_+ &= v_{+,+} + v_{+,-}. \end{aligned}$$

Consider the \mathbb{Z} -module, or free abelian group, M from Section 2.1, generated by basis $\mathcal{B} = \{a, b\}^{\mathbb{N}}$, and the faithful representation $\pi_T: L_{2,\mathbb{Z}} \rightarrow \text{End}(M)$. Let

$$f = \pi_T(u_{+,+}), g = \pi_T(u_{+,-}), s = \pi_T(v_{+,+}) \text{ and } t = \pi_T(v_{+,-})$$

For (2) first note that f, g, s, t satisfy the hypotheses (i), (ii) of Lemma 4.7. Moreover, we have

$$[f - g, s - t] = [\pi_T(u), \pi_T(v)] = \pi_T([u, v]) = 0.$$

Lemma 4.7 now implies that

$$\pi_T([u_+, v_+]) = [\pi_T(u_+), \pi_T(v_+)] = [f + g, s + t] = 0.$$

Hence $[u_+, v_+] = 0$, and (2) holds. Property (3) holds because (2) gives hypothesis (iii) from Lemma 4.8, and then Lemma 4.8 gives the result. \square

Proof of Theorem 4.1. We know from Proposition 3.1 that $L_{\mathbb{Z}}[w, w^{-1}, z, z^{-1}]$ embeds into $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ by a unital $*$ -homomorphism. We claim that there is *no* unital $*$ -homomorphic embedding of $L_{\mathbb{Z}}[w, w^{-1}, z, z^{-1}]$ into $L_{2,\mathbb{Z}}$. The result will then follow from this claim because any such embedding $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \hookrightarrow L_{2,\mathbb{Z}}$ would induce an embedding $L_{\mathbb{Z}}[w, w^{-1}, z, z^{-1}] \hookrightarrow L_{2,\mathbb{Z}}$, contradicting the claim.

To see why the claim holds, let $\phi: L_{\mathbb{Z}}[w, w^{-1}, z, z^{-1}] \rightarrow L_{2,\mathbb{Z}}$ be a unital $*$ -homomorphism. Then $u = \phi(w)$ and $v = \phi(z)$ are commuting unitaries in $L_{2,\mathbb{Z}}$. We know from parts (1) and (2) of Proposition 4.6 that u_+ and v_+ are then commuting unitaries. Since $u_+, v_+ \in \mathcal{U}_V$, we know from Proposition 3.4 that there is a nonzero polynomial $q(w, z) \in \mathbb{Z}[w, z]$ with $q(u_+, v_+) = 0$. Now (3) of Proposition 4.6 says there is a nonzero polynomial $\tilde{q}(w, z) \in L_{\mathbb{Z}}[w, w^{-1}, z, z^{-1}]$ with $\tilde{q}(u, v) = 0$. So $\phi(\tilde{q}(w, z)) = \tilde{q}(u, v) = 0$, and hence ϕ is not injective. This proves the claim. \square

5. THE CASE $L_{2,R} \otimes L_{2,R}$

In this section we make some remarks about what our results indicate about the general question of whether $L_{2,R} \otimes L_{2,R}$ embeds into $L_{2,R}$. While we cannot answer the question completely we can give some partial results.

Our result for $R = \mathbb{Z}$ depends on Proposition 4.4, which gives us a nice characterisation of the unitaries in $L_{2,\mathbb{Z}}$. Since the difference in the characterisations of a general unitary and a unitary in \mathcal{U}_V is so minor (the allowance for minus signs in the former), Proposition 4.6 allows us to bump up the non-embedding result Theorem 3.3 which uses \mathcal{U}_V to the argument required to get a non-embedding $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \not\hookrightarrow L_{2,\mathbb{Z}}$. In the general setting we lose such tight control on the unitaries as the following example illustrates.

Example 5.1. *Consider the field $K = \mathbb{R}$. There is a unital embedding of $M_2(\mathbb{R})$ into $L_{2,\mathbb{R}}$ such that*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \mapsto \lambda_{11}aa^* + \lambda_{12}ab^* + \lambda_{21}ba^* + \lambda_{22}bb^*.$$

So for any angle θ we can map the rotation unitary for θ into $L_{2,\mathbb{R}}$ to get a unitary

$$(\cos \theta)aa^* - (\sin \theta)ab^* + (\sin \theta)ba^* + (\cos \theta)bb^* \in L_{2,\mathbb{R}}.$$

We immediately see that the coefficients of unitaries in $L_{2,\mathbb{R}}$ can take uncountably many values. Furthermore, we cannot simply throw away the coefficients, like in Proposition 4.6, since $aa^ + ba^* + ab^* + bb^*$ is not a unitary in $L_{2,\mathbb{R}}$.*

While we cannot see how to generalise our techniques beyond the case $R = \mathbb{Z}$, we can still apply our results to prove results about more general rings. Namely, we can show that if K is a field of characteristic 0, then any potential embedding of $L_{2,K} \otimes L_{2,K}$ into $L_{2,K}$ must use the coefficients.

To make the claim precise we define

$$L_{2,R}^1 = \left\{ x \in L_{2,R} : x = \sum_{i=1}^n \alpha_i \beta_i^* \text{ for some distinct pairs } \alpha_i, \beta_i \in \{a, b\}^* \right\}.$$

That is, $L_{2,R}^1$ consists of the elements of $L_{2,R}$ that can be written without coefficients.

Proposition 5.2. *Let K be a field and let $\phi: L_{2,K} \otimes L_{2,K} \rightarrow L_{2,K}$ be a unital $*$ -homomorphism. If K has characteristic 0, then at least one of $\phi(a \otimes 1), \phi(b \otimes 1), \phi(1 \otimes a)$, and $\phi(1 \otimes b)$ will not be in $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$.*

Proof. Because K has characteristic 0 we can, by the Cuntz-Krieger Uniqueness Theorem [21, Theorem 6.5], embed $L_{2,\mathbb{Z}}$ into $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$ (note that $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$ is a \mathbb{Z} -algebra). As a \mathbb{Z} -module $L_{2,K}$ is torsion free, since K has characteristic 0, and hence it is flat as a \mathbb{Z} -module. Therefore we can use the observation from Remark 3.2 to see that we also get an embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,K} \otimes L_{2,K}$ by simply tensoring the embedding of $L_{2,\mathbb{Z}}$ into $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$ with itself.

If all four of $\phi(a \otimes 1), \phi(b \otimes 1), \phi(1 \otimes a)$, and $\phi(1 \otimes b)$ are in $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$, then we can define a unital $*$ -homomorphism $\psi: L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \rightarrow L_{2,\mathbb{Z}}$ such that

$$\begin{aligned} \psi(a \otimes 1) &= \phi(a \otimes 1), & \psi(b \otimes 1) &= \phi(b \otimes 1), \\ \psi(1 \otimes a) &= \phi(1 \otimes a), & \psi(1 \otimes b) &= \phi(1 \otimes b). \end{aligned}$$

By construction ψ is the restriction of ϕ when we think of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ as sitting inside $L_{2,K} \otimes L_{2,K}$. Since $L_{2,K} \otimes L_{2,K}$ is simple, ϕ is necessarily injective, and therefore ψ is injective. But that contradicts Theorem 4.1, so at least one of $\phi(a \otimes 1), \phi(b \otimes 1), \phi(1 \otimes a)$ and $\phi(1 \otimes b)$ will not be in $\text{span}_{\mathbb{Z}}(L_{2,K}^1)$. \square

We find this result to be interesting since it indicates that if $L_{2,K} \otimes L_{2,K}$ embeds into $L_{2,K}$ for some field of characteristic 0, then the embedding must somehow reference the field.

Our results also allows us to say something about potential embeddings for general rings. We say a unitary in an involutive R -algebra has *full spectrum* if $q(\alpha) \neq 0$ for all nonzero $q \in R[x]$. (Here, the element $q(\alpha)$ means the obvious thing. See [12, Remark 2.6] for the details.)

Proposition 5.3. *Let R be a commutative ring with unit. Suppose that $\phi: L_{2,R} \otimes L_{2,R} \rightarrow L_{2,R}$ is an injective, unital $*$ -homomorphism. If $u, v \in \mathcal{U}(L_{2,R})$ have full spectrum then $\phi(u \otimes 1)$ and $\phi(1 \otimes v)$ cannot both be in \mathcal{U}_V .*

Proof. By Proposition 3.1 and its proof we see that $z \mapsto u \otimes 1$ and $w \mapsto 1 \otimes v$ defines an injective $*$ -homomorphism from $L_R[z, z^{-1}, w, w^{-1}]$ to $L_{2,R} \otimes L_{2,R}$. Call it ψ . Then $\phi \circ \psi$ is an injective $*$ -homomorphism from $L_R[z, z^{-1}, w, w^{-1}]$ to $L_{2,R}$, and so by Theorem 3.3 we cannot have that both $\phi(\psi(z)) = \phi(u \otimes 1)$ and $\phi(\psi(w)) = \phi(1 \otimes v)$ are in \mathcal{U}_V . \square

Again the result gives an indication that the ring of coefficients is likely to matter to a given embedding. However, when working with rings that do not have characteristic 0 we do not have that $\mathcal{U}_V = \mathcal{U}_1$ (Example 2.2) and so exactly how the coefficients matter is murky.

6. NOTE ADDED IN PROOF

After an initial version of this manuscript appeared on the arXiv, the second named author and Rune Johansen used the characterization of unitaries in $L_{2,\mathbb{Z}}$ given in Proposition 4.4 to give a similar characterization of the projections. We can use this to conclude that not only is there no *unital* $*$ -homomorphic embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,\mathbb{Z}}$ (Theorem 4.1) there is no $*$ -homomorphic embedding.

Proposition 6.1. *All nonzero projections in $L_{2,\mathbb{Z}}$ are Murray-von Neumann equivalent to the unit.*

Proof. Let $p \in L_{2,\mathbb{Z}}$ be a projection. By [17, Theorem 4.5] there exist paths $\beta_1, \beta_2, \dots, \beta_n \in \{a, b\}^*$ with disjoint cylinder sets such that

$$p = \sum_{i=1}^n \beta_i \beta_i^*.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be paths with disjoint cylinder sets whose union is $\{a, b\}^{\mathbb{N}}$. Then $t_i = \alpha_i \beta_i^*$ shows that $\alpha_i \sim \beta_i$, for $i = 1, 2, \dots, n$. Since $\{\beta_i \beta_i^*\}$ and $\{\alpha_i \alpha_i^*\}$ are two collections of pairwise orthogonal projections, we then get (see [9, Proposition 4.2.4]) that

$$p = \sum_{i=1}^n \beta_i \beta_i^* \sim \sum_{i=1}^n \alpha_i \alpha_i^* = 1. \quad \square$$

Corollary 6.2. *There is no $*$ -algebraic embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,\mathbb{Z}}$.*

Proof. Let $\phi: L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \rightarrow L_{2,\mathbb{Z}}$ be a nonzero $*$ -homomorphism and let $p = \phi(1 \otimes 1)$. By Proposition 6.1 we have $p \sim 1$, so we can pick a $t \in L_{2,\mathbb{Z}}$ such that $t^* t = 1$ and $t t^* = \phi(1 \otimes 1)$. Then we can define a $*$ -homomorphism $\psi: L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \rightarrow L_{2,\mathbb{Z}}$ by $\psi(x) = t^* \phi(x) t$. Note that ψ is unital and that ψ is an embedding if and only if ϕ is. But by Theorem 4.1, ψ cannot be an embedding and hence ϕ is not an embedding. \square

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